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Regular Maps on Non-orientable Surfaces

MARSTON CONDER and BRENT EVERITT
Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

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Abstract. It is well known that regular maps exist on the projective plane but not on the Klein bottle, nor the non-orientable surface of genus 3. In this paper several infinite families of regular maps are constructed to show that such maps exist on non-orientable surfaces of over 77 per cent of all possible genera.

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1. Introduction

Regular maps may be thought of as a combinatorial generalization of the Platonic solids. The standard authority is the book by Coxeter and Moser [4], in which copious examples are provided of regular maps that lie on orientable surfaces. In fact, it is easy to construct a regular map on an orientable surface of any genus $g \geq 0$.

The situation where the map resides on a non-orientable surface is also considered briefly in [4]. It is easily shown how to obtain a regular map on the projective plane: just by identifying antipodal points of some of the Platonic solids for instance. Less obvious, however, is the non-existence of regular maps on the Klein bottle (genus 2) or on the non-orientable surface of genus 3.

Despite this apparent evidence to the contrary, regular maps do exist on *most* non-orientable surfaces. (This statement will be clarified later.) Table 8 of [4] contains many examples, and in [6] Steve Wilson gives an infinite family of non-orientable surfaces that admit regular maps whose underlying graph is complete in each case.

In this paper we describe a group-theoretic construction which we then use to show that regular maps exist on non-orientable surfaces of over 77 per cent of all possible genera. Our construction involves taking the semi-direct product of a cyclic group N (of variable order) by a fixed group H which is known to be the automorphism group of a non-orientable regular map M with appropriate parameters; this is achieved in such a way that the resulting groups are the automorphism groups of a family of non-orientable maps which all have the given map M as a quotient.

First we describe some of the standard background on regular maps in Section 2, including the connection with group presentations which is fundamental to our approach. Details of the construction and a family of starting blocks (that is, possibilities for the 'top' group H) are given in Section 3, and the genera obtained so far are summarized and discussed in Section 4.

2. Basic Concepts and the Groups $G^{p,q,r}$

A *map* is a 2-cell embedding of a connected graph into a closed surface without boundary. Such a map M is thus composed of a vertex-set V, an edge-set E, and a set of faces which we will denote by F. The faces of M are of course the connected components of the space obtained by removing the embedded graph from the surface; alternatively, in the orientable case, they can be considered without recourse to geometry by considering just the underlying graph together with a 'rotation' at each vertex (see [1]).

Associated also with any map is a set of *darts*, or *arcs*, which are the incident vertex-edge pairs $(v, e) \in V \times E$. Each dart is made up of two *blades*, one corresponding to each face containing the edge e (except in degenerate situations where an edge lies in just one face, but these will not concern us here). An *automorphism* of a map M is a permutation of its blades, preserving the properties of incidence, and as usual these form a group under composition, called the *automorphism group* of the map, and denoted by $\mathrm{Aut}(M)$.

Now if the group $\mathrm{Aut}(M)$ contains elements R and S with the property that R cyclically permutes the consecutive edges of some face f (in single steps around f), and S cyclically permutes the consecutive edge incident to some vertex v of f (in single steps around v), then following Wilson [6] we call M a *rotary* map. In this case (by connectedness) the group $\mathrm{Aut}(M)$ acts transitively on the vertices, on edges, and on faces of M, and it follows that all the faces are bordered by the same number of edges, say p, while all the vertices have the same degree, say q.

Notice also that the automorphism RS interchanges the vertex v with one of its neighbours along an edge e (on the border of f), interchanging f with the other face containing e in the process. The three automorphisms R, S and RS may thus be viewed as rotations which satisfy the relations

$$R^p = S^q = (RS)^2 = 1.$$

If a rotary map M admits also an automorphism a which (like RS) 'flips' the edge e but (unlike RS) preserves the face f, then we say the map M is *regular*. This automorphism a is essentially a reflection, about an axis passing through the midpoints of the edge e and the face f. Similarly, the automorphisms $b = aR$ and $c = bS$ may also be thought of as reflections, and the following relations are satisfied:

$$a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^2 = 1.$$

Note: In [4] rotaty maps are called regular, while regular maps are called reflexible; this notation appears to have been abandoned.

By connectedness, every automorphism is uniquely determined by its effect on any blade, and in particular, the stabilizer of every blade is trivial. In the case of a rotary map M, the group Aut(M) is transitive on darts, but if M is regular, then Aut(M) is transitive (indeed regular) on blades. As a consequence, the automorphism group of a regular map M is generated by the reflections a, b, c, described above.

Counting the number of blades containing a given edge e yields $|\mathrm{Aut}(M)| = 2|E|$ when M is rotary but not regular, while $|\mathrm{Aut}(M)| = 4|E|$ when M is regular. In either case, counting the number of darts incident with a given vertex, edge or face gives the well-known identity $q|V| = 2|E| = p|F|$.

One other geometric notion we consider is that of a *Petrie polygon*: this is a 'zig zag' circuit in which every two but no three consecutive edges border the same face. It is not difficult to see that every edge of a map M lies in exactly two Petrie polygons, and so we can talk of the *Petrie dual* $P(M)$ of M as that map which has the same vertices and edges as M but whose faces are the Petrie polygons of M.

Note that $P(M)$ is dual to M in the sense that it establishes a duality between faces and Petrie polygons: the Petrie polygons of $P(M)$ are the faces of M, and vice-versa. Also, as with the standard dual $D(M)$ (obtained by interchanging the roles of the vertices and faces), the Petrie dual $P(M)$ is regular if and only if M is regular; see Wilson [5]. On the other hand, unlike the standard dual, the Petrie dual may lie on a different surface from that of the given map M, and in particular, $P(M)$ can often be orientable when M is not.

When M is regular, the automorphism abc acts like a glide reflection, moving some Petrie polygon along itself in consecutive steps. If the Petrie polygon contains r edges, then a, b and c satisfy the relations

$$a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^2 = (abc)^r = 1 \tag{1}$$

and we say that M has parameters $\{p, q\}_r$. Note that $D(M)$ then has parameters $\{q, p\}_r$, while $P(M)$ has parameters $\{r, q\}_p$, and in fact all six permutations of $\{p, q, r\}$ are possible; see [5].

The relations (1) are defining relations for the abstract group $G^{p,q,r}$ defined by Coxeter (see [3]). By what we have outlined above, the automorphism group of any regular map M of type $\{p, q\}_r$ is a non-degenerate homomorphic image of this group $G^{p,q,r}$, 'non-degenerate' meaning that the orders of a, b and c and their products ab, bc and ca are preserved.

Conversely, any non-degenerate homomorphism from $G^{p,q,r}$ to a finite group G yields a regular map M with parameters $\{p, q\}_r$ on which G acts as its automorphism group. In fact the vertices, edges and faces of M may be taken as the cosets in G of the (images of the) subgroups $V = \langle b, c\rangle$, $E = \langle a, c\rangle$ and $F = \langle a, b\rangle$ respectively, with G acting by right multiplication. (These three subgroups then become the stabilizers of some mutually incident vertex v, edge e and face f respectively.)

Finally, note that the regular map M is orientable if and only if the subgroup $\langle ab, bc\rangle$ of $G = \mathrm{Aut}(M)$ has index 2 in G (as does its pre-image in the group $G^{p,q,r}$). In this case $\langle ab, bc\rangle = \langle R, S\rangle$ has two orbits on blades, with the two blades associated with any given dart lying in different orbits. The elements of this even-length word subgroup are 'rotations', while all other elements of G are 'reflections' (or glide reflections), reversing the orientation of the surface.

In the non-orientable case, however, the subgroup $\langle ab, bc\rangle$ has index 1 in G. In particular, this subgroup has just one orbit on blades – as shown in [4] using a simple colouring argument – and there are no true reflections: every reflection is a product of rotations. The genus of M is defined as the genus g of the non-orientable surface on which M is embedded, and is given the usual formula in terms of the Euler characteristic: $2 - g = \chi = |V| - |E| + |F|$. As $2p|F| = 4|E| = 2q|V| = |G|$, the latter formula simplifies to $g = |G|(1/4 - 1/2p - 1/2q) + 2$.

3. The Construction

Using the correspondence between regular maps and generators for their automorphism groups (as described in the previous section), we can set about finding regular maps on non-orientable surfaces by constructing groups with the appropriate properties.

Suppose H is a finite group generated by elements u, v and t which satisfy the relations $u^2 = v^q = (uv)^p = t^2 = (ut)^2 = (vt)^2 = 1$, with v and uv having (true) orders p and q respectively, and such that uv and v^2 generate a subgroup of index 2 in H, containing t. In this case necessarily q is even, and H itself is generated by u and v. Also let N be a cyclic group of order n, generated by some element w.

Now form the semi-direct product (or split extension) NH of N by H, with H acting on N by conjugation in such a way that u and v both invert w, this is: $u^{-1}wu = w^{-1} = v^{-1}wv$. In this group define $a = wut, b = tv$ and $c = t$, and consider the subgroup G generated by a, b and c.

Note that w is centralized by both uv and v^2, and therefore also by t. In particular, this implies $a^2 = (wut)^2 = ww^{-1}(ut)^2 = 1$, so that a has order 2. Similarly b, c and ca all have order 2, while $bc = v^{-1}$ has order q. Next $ab = wut^2v = wuv$ and therefore $(ab)^p = w^p$, hence the order of ab is equal to $\mathrm{lcm}(n, p)$.

It follows that G is the automorphism group of some regular map M whose vertices all have degree q and whose faces are all bounded by $\mathrm{lcm}(n, p)$ edges. The numbers of vertices, edges and faces of M depend on the order of G, and its orientability (or otherwise) depends on whether or not the subgroup generated by ab, bc and ca has index 2 (or 1).

To see what can happen, we consider a few examples:

EXAMPLE 3.1. Let H be the symmetric group S_4, and in this group let $u = (1,2), v = (1,2,3,4)$ and $t = (1,2)(3,4)$. Note that u and v themselves generate S_4, and that $t = (uv)^{-1}v^2uv$; in fact that subgroup generated by $uv = (1,3,4)$ and $v^2 = (1,3)(2,4)$ is the alternating group A_4, which obviously contains t, so the above conditions are satisfied.

Now if w has order n, then ab has order n if n is divisible by 3, and $3n$ otherwise. Also since $u^2 = v^4 = (uv)^3 = 1$ are defining relations for the group $H = S_4$, it follows that $G = \langle a,b,c\rangle$ has order 24 times the order of w^3, that is: $|G| = 8n$ if n is divisible by 3, while $|G| = 24n$ if n is not.

Further, as $(ab)^{-1}(cb)^2ab = (wuv)^{-1}v^2wuv = (uv)^{-1}v^2uv = t = c$ we find c lies in the rotation subgroup $\langle ab, bc\rangle$, which therefore has index 1 in G. It follows that the corresponding map is non-orientable. By the formula given in Section 2, the genus of this regular map is $8n(1/4-1/2n-1/8)+2 = n-2$ if n is divisible by 3, and $24n(1/4-1/6n-1/8)+2 = 3n-2$ otherwise.

In either case, we obtain a family of non-orientable regular maps of genera $3k-2$, for $k = 1,2,3,\ldots$, as given also in [6]. This accounts for one-third of all positive integers.

EXAMPLE 3.2. Let H be the subgroup of S_6 generated by the three permutations $u = (1,4)(2,3)(5,6), v = (1,2,3,4,5,6)$ and $t = (2,6)(3,5)$. Note that $\langle u,v\rangle$ is dihedral of order 12, and that $t = uv^3$. The subgroup generated by $uv = (1,5)(2,4)$ and $v^2 = (1,3,5)(2,4,6)$ is dihedral of order 6, and contains t, so the required conditions are fulfilled.

In this case if w has order n, then ab has order $2n$ if n is odd, but n if n is even. Also since $u^2 = v^6 = (uv)^2 = 1$ are defining relations for the dihedral group H, it follows that $G = \langle a,b,c\rangle$ has order 12 times the order of w^2, that is: $|G| = 12n$ if n is odd, while $|G| = 6n$ if n is even.

Further, since $uvt = v^2$ has order 3, we find that order of $abc = wuvt$ is equal to $\mathrm{lcm}(n,3)$. In particular, if n is odd then $(abc)^{3n} = 1$ is a relation of odd length in the generators a,b,c of G, and hence the rotation subgroup $\langle ab, bc\rangle$ has index 1 in G, and the corresponding map is non-orientable. By the given formula, the genus of this regular map is $12n(1/4-1/4n-1/12)+2 = 2n-1$. (On the other hand, if n is even, the defining relations for H show the rotation subgroup has index 2 in G, so the corresponding map is orientable in that case.)

Letting $n = 2k-1$, we thus obtain a family of non-orientable regular maps of genera $4k-3$, for $k = 1,2,3,\ldots$. This accounts for 25 per cent of all positive integers, and when taken together with the family obtained in Example 3.1, accounts for exactly half: all $g \equiv 1,4,5,7,9$ or $10 \bmod 12$.

EXAMPLE 3.3. Let H be the subgroup of S_7 generated by the three permutations $u = (1,2)(4,5)(6,7), v = (2,3,4)(6,7)$ and $t = (1,5)(2,4)$. In this case H is isomorphic to $A_5 \times C_2$, with $uv = (1,3,4,5,2)$ and $v^2 = (2,4,3)$ generating the subgroup A_5, and $v^3 = (6,7)$ generating the centre of order 2. The element t is not only contained in the former subgroup, but is also expressible as a product of conjugates of v^2.

When we form the semi-direct product NH, the element ab has order n if n is divisible by 5, and $5n$ otherwise. On the other hand, as $(abcbcb)^2 = (wuv^3)^2 = w^2$ we find $G = \langle a, b, c \rangle = NH$, and therefore the group G has order $120n$ for every n.

The corresponding map is non-orientable for all n (as in Example 3.1), since the element c is expressible as a product of conjugates of $(cb)^2$. The genus is $20n - 58$ if n is divisible by 5, and $20n - 10$ otherwise, and thus we have a family of non-orientable regular maps of genus g for all $g \equiv 10, 30, 42, 50$ or 70 mod 100.

Similar arguments are required for other examples. In most cases it is easy to determine exactly when the rotation subgroup $\langle ab, bc \rangle$ has index 1 (by considering either conjugates of v or v^2, or the order of the product abc, for example). The difficulty lies in the calculation of the order of G, which largely depends on relations satisfied by the generators of the original group H. For permutation groups, however, this may be achieved by inspection or with the help of the 'relations' command in the CAYLEY system [2].

In Table 1 we list a number of examples which may be used as the 'top' group H in our-semi-direct product construction. Each item in the table yields an infinite family of non-orientable regular maps, all of which have the map corresponding to H (obtainable in the case $n = 1$) as a quotient. We provide generating permutations for H, state whether the construction works for all n or just for n odd, and describe the parameters of the regular map(s) which result, along with their genera.

Many of these examples were found by searching for suitable permutation representations of the groups $[p, q] = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^2 = 1 \rangle$ for small p and q, using the 'low index subgroups' algorithm (available in the CAYLEY system [2]). Of course many other examples may be found, however their contribution to the genus spectrum will become less significant as their orders (and the parameters p and q) increase.

4. Results and Discussion

The families of non-orientable regular maps presented in Section 3 account for over 77 per cent of all genera. To partly see how this comes about, note that the first two examples listed in Table 1 cover 6 of the 12 residue classes modulo 12, the first three cover 26 of the 48 residue classes modulo 48, and so on. In fact our families account for 29 547 540 of all residue classes modulo 38 102 400 ($=2^7 3^5 5^2 7^2$), a little over 77.5 per cent of all positive integers.

There are, of course, many sporadic examples with odd parameters, for example: $G^{3,7,9} \cong \mathrm{PSL}(2, 8)$, of order 504, yields a regular map of non-orientable genus 8, while $G^{3,7,13} \cong \mathrm{PSL}(2, 13)$, of order 1092, gives one of non-orientable genus 15 whose Petrie dual is also non-orientable and of genus 51 (see Table 8 in [4]), and finally $G^{5,5,5} \cong \mathrm{PSL}(2, 11)$ of order 660 yields a regular map of non-orientable genus 35.

TABLE I.

Generating permutations for H	Order of H	Genera g
$u=(1,2)$ $v=(1,2,3,4)$ $t=(1,2)(3,4)$ all n type $\{3k,4\}$	24	$g\equiv 1 \bmod 3$
$u=(1,4)(2,3)(5,6)$ $v=(1,2,3,4,5,6)$ $t=(2,6)(3,5)$ n odd type $\{2n,6\}$	12	$g\equiv 1 \bmod 4$
$u=(1,2)(5,6)$ $v=(2,3,4)(5,6)$ $t=(3,4)$ n odd type $\{4n,6\}$	48	$g\equiv 4 \bmod 16$
$u=(1,2)(3,5)(6,7)(8,10)$ $v=(2,3,7,8)(4,5,9,10)$ $t=(3,8)(5,10)$ all n type $\{5k,4\}$	160	$g\equiv 6 \bmod 20$
$u=(1,2)(4,8)(5,7)$ $v=(1,2,3,4,5,6)(7,8,9)$ $t=(1,2)(3,6)(4,5)(7,8)$ n odd type $\{6k,6\}$	108	$g\equiv 11 \bmod 36$
$u=(3,5)(4,7)(6,10)(8,11)(9,13)(12,14)$ $v=(1,2,4,8,10,14,16,15,13,11,6,3)(5,7,12,9)$ $t=(1,2)(3,4)(5,7)(6,8)(9,12)(10,11)(13,14)(15,16)$ n odd type $\{8n,12\}$	96	$g\equiv 16 \bmod 40$
$u=(1,2)(5,9)(6,7)(8,10)(11,12)$ $v=(1,2,4,6,10,12,9,7,11,8,5,3)$ $t=(1,2)(3,4)(5,6)(7,9)(8,10)(11,12)$ n odd type $\{6k,12\}$	144	$g\equiv 20 \bmod 60$
$u=(2,5)(4,9)(6,14)(7,12)(8,15)(10,19)(11,21)$ $(13,24)(16,25)(17,28)(20,23)(22,30)(26,29)(31,32)$ $v=(1,2,6,15,27,20,10,4)(3,8,17,21,18,9,16,7)$ $(5,11,22,30,23,12,24,13)(14,25,31,29,19,28,32,26)$ $t=(1,3)(2,7)(4,8)(5,12)(6,16)(9,15)(10,17)(11,23)(13,24)$ $(14,25)(18,27)(19,28)(20,21)(22,30)(26,31)(29,32)$ all n type $\{5k,8\}$	320	$g\equiv 30 \bmod 60$

TABLE I. – *continued*

Generating permutations for H	Order of H	Genera g
$u = (2,5)(3,4)(8,11)(9,10)$ $v = (1,2,3)(4,5,7,9,8,6)(10,12,11)$ $t = (2,3)(4,5)(6,7)(8,9)(10,11)$ n odd type $\{8n, 6\}$	192	$g \equiv 22 \bmod 64$
$u = (2,3)(4,5)(6,7)$ $v = (1,2,3,4,5)(6,7)$ $t = (2,5)(3,4)$ all n type $\{3k, 10\}$	120	$g \equiv 6, 14, 30 \bmod 72$
$u = (3,4)(5,6)$ $v = (1,2,4,6,8,7,5,3)$ $t = (1,2)(3,4)(5,6)(7,8)$ all n type $\{6k, 8\}$	384	$g \equiv 42 \bmod 72$
$u = (1,3)(2,7)(4,5)(6,9)(8,10)$ $v = (1,2)(3,5)(4,6,8)(7,10,9)$ $t = (1,3)(2,5)(4,7)(6,9)(8,10)$ n odd type $\{4n, 6\}$	240	$g \equiv 12 \bmod 80$
$u = (4,5)(6,7)$ $v = (1,2,4,6,8,10,9,7,5,3)$ $t = (2,3)(4,5)(6,7)(8,9)$ all n type $\{4k, 10\}$	240	$g \equiv 20, 38 \bmod 96$
$u = (1,2)(4,5)(6,7)$ $v = (2,3,4)(6,7)$ $t = (1,5)(2,4)$ all n type $\{5k, 6\}$	120	$g \equiv 10, 30, 42, 50, 70 \bmod 100$
$u = (1,2)(3,6)(4,7)(5,8)(9,11)(10,12)(13,14)$ $v = (1,3)(2,4,6,5)(7,9)(8,10)(11,13,12,14)$ $t = (4,5)(7,8)(9,10)(11,12)$ all n type $\{7k, 4\}$	896	$g \equiv 50 \bmod 112$
$u = (2,4)(3,5)(6,7)$ $v = (1,2,3,4,5)(6,7)$ $t = (2,5)(3,4)$ all n type $\{5k, 10\}$	120	$g \equiv 14, 38, 62, 86 \bmod 120$

TABLE I. – *continued*

Generating permutations for H	Order of H	Genera g
$u=(1,2)(3,4)(7,8)(9,10)$ $v=(1,2,4,6,8,10,9,7,5,3)$ $t=(1,2)(3,4)(5,6)(7,8)(9,10)$ n odd type $\{4n, 10\}$	320	$g \equiv 26 \bmod 128$
$u=(1,2)(3,7)(4,8)(5,9)(6,10)(11,13)(12,14)$ $(15,16)(17,18)$ $v=(1,3,6,2,5,4)(7,11,15,18,14,10)$ $(8,9,13,17,16,12)$ $t=(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$ $(17,18)$ n odd type $\{12k, 6\}$	432	$g \equiv 56 \bmod 144$
$u=(1,2)(3,5)(6,7)$ $v=(2,3,4,5)(6,7)$ $t=(3,5)$ n odd type $\{6k, 4\}$	240	$g \equiv 12, 32, 132 \bmod 180$
$u=(3,4)(5,7)(6,8)$ $v=(1,2,4,6,8,7,5,3)$ $t=(1,2)(3,4)(5,6)(7,8)$ all n type $\{3k, 8\}$	336	$g \equiv 9, 23, 72 \bmod 189$
$u=(3,5)(4,7)(6,10)(8,13)(9,15)(12,16)$ $v=(1,2,4,8,14,15,13,10,16,11,6,3)$ $(5,7,12,9)$ $t=(1,2)(3,4)(5,7)(6,8)(9,12)(10,13)$ $(11,14)(15,16)$ n odd type $\{12k, 12\}$	576	$g \equiv 98 \bmod 240$
$u=(3,5)(4,6)(7,10)(8,12)(9,11)$ $v=(1,2,4,3)(5,7,11,8)(6,9,12,10)$ $t=(1,2)(3,4)(5,6)(7,10)(8,9)(11,12)$ all n type $\{9k, 4\}$	648	$g \equiv 47, 128, 137 \bmod 243$
$u=(3,5)(4,7)(6,8)$ $v=(1,2,4,8,7,5,6,3)$ $t=(1,2)(3,4)(5,7)(6,8)$ all n type $\{4k, 8\}$	336	$g \equiv 23, 44, 86, 149 \bmod 252$
$u=(2,5)(4,8)(6,10)(7,11)(9,13)(12,15)$ $v=(1,2,6,4)(3,7,9,5)(8,10,14,12)$ $(11,15,16,13)$ $t=(1,3)(2,5)(4,7)(6,9)(8,11)(10,13)$ $(12,15)(14,16)$ all n type $\{6k, 4\}$	672	$g \equiv 30, 86, 114 \bmod 252$

TABLE I. – *continued*

Generating permutations for H	Order of H	Genera g
$u = (1,3)(4,11)(6,13)(7,8)(9,10)(14,16)$ $v = (1,2,6,4)(3,7,9,5)(8,10,14,12)$ $(11,15,16,13)$ $t = (1,3)(2,5)(4,7)(6,9)(8,11)(10,13)$ $(12,15)(14,16)$ all n type $\{8k,4\}$	672	$g \equiv 44, 86, 170, 212 \bmod 336$
$u = (3,5)(4,7)(6,8)$ $v = (1,2,4,8,6,3)$ $t = (1,2)(3,4)(5,7)(6,8)$ all n type $\{7k,6\}$	336	$g \equiv 34, 90, 146, 202, 226, 258,$ $314 \bmod 392$
$u = (1,2)(5,8)(6,7)$ $v = (1,2,4,6,8,7,5,3)$ $t = (1,2)(3,4)(5,6)(7,8)$ all n type $\{7k,8\}$	336	$g \equiv 41, 104, 167, 230, 275, 293,$ $356 \bmod 441$
$u = (1,2)(3,4)(5,8)(6,10)(7,9)$ $v = (1,10,9,3,7,2,5,6,8,4)$ $t = (3,6)(4,10)(5,7)(8,9)$ all n type $\{5k,10\}$	720	$g \equiv 74, 218, 362, 506 \bmod 720$

Every positive integer $g \leq 100$ other than $2, 3, 18, 24, 27, 39, 48, 54, 59, 60, 63, 71, 75, 87, 95$ and 99 is known to be the genus of some non-orientable regular map. Of the exceptions, non-orientable surfaces of genus 2 or 3 are definitely known not to admit regular maps (see [4]), and we believe 18, 24 and 27 have also been eliminated by Steve Wilson (private communication).

What of the remaining genera? As our method requires (at least) one of the parameters to be even, its scope is limited. As well as this, examples larger than those we have provided will produce relatively fewer new possibilities, and so it would seem unlikely that our method will yield much more than it already has – but who knows? The true picture is still far from clear.

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References

1. Biggs, N. L. and White, A. T.: *Permutation Groups and Combinatorial Structures*, L.M.S. Lecture Note Series, vol. 33, Cambridge University Press, 1979.
2. Cannon, J. J.: An introduction to the group theory language CAYLEY, in M. Atkinson (ed.), *Computational Group Theory*, Academic Press, San Diego, London, 1984, pp. 145–183.
3. Coxeter, H. S. M.: The abstract groups $G^{m,n,p}$, *Trans. Amer. Math. Soc.* **45** (1939), 73–150.
4. Coxeter, H. S. M. and Moser, W. O. J.: *Generators and Relations for Discrete Groups*, 4th edn, Springer-Verlag, Berlin, 1980.
5. Wilson, S. E.: Operators over regular maps, *Pacific J. Math.* **81** (1979), 559–568.
6. Wilson, S. E.: Cantankerous maps and rotary embeddings of K_n, *J. Combin. Theory Series B* **47** (1989), 262–273.